ON THE NATURE OF THE BOUNDARY LAYER IN STAGNATION POINT FLOW

By

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To My Wife
Mary
For All Her Patience
And Understanding

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LIST OF SYMBOLS

a_i	Constants defined in Chapter IV
b_i	Constants defined in Chapter III
A_{kl}	Constants defined in Chapter IV
B_{kl}	Constants defined in Chapter IV
C_{kl}	Constants defined in Chapter IV
f	Similarity function of boundary layer solution
g	Function defined in Chapter II
h	Parameter of boundary layer solution
i	Imaginary number (0,1)
p	Dimensionless hydrostatic pressure
$\bar{p}$	Hydrostatic pressure
r	Polar coordinate
R_n	Reynolds number
u	Dimensionless velocity parallel to the wall
$\bar{u}$	Velocity parallel to the wall
U	Reference velocity
v	Dimensionless velocity perpendicular to the wall
$\bar{v}$	Velocity perpendicular to the wall
x	Dimensionless distance along the wall
$\bar{x}$	Distance along the wall
y	Dimensionless distance perpendicular to the wall

$\bar{y}$	Distance perpendicular to the wall
z	Coordinate defined in Chapter I
z^*	Complex conjugate of z
δ	Jet thickness
η	Similarity coordinate of boundary layer solution
θ	Polar coordinate
μ	Viscosity
ν	Kinematic viscosity
ρ	Density
τ_w	Shear stress at the wall
φ	Function defined in Chapter III
χ	Function defined in Chapter III
ψ	Stream function for boundary layer flow
ψ_{ij}	Stream function for Stokes flow

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ON THE NATURE OF THE BOUNDARY LAYER IN STAGNATION POINT FLOW

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The flow at and in the neighborhood of a stagnation point is obtained from a known solution of the boundary layer flow downstream. The flow is assumed to impinge on a flat plate at right angles. The solution is obtained using Stokes' approximations, and is obtained in terms of arbitrary constants. These constants are evaluated by requiring the shear stress at the wall for the Stokes flow to match that of the boundary layer flow.

Graphs are obtained by matching the two solutions at 1.0, 5.0, 10.0, and 14.3 cm. from the stagnation point. A graph is also obtained for the variation of the shear stress at the wall with distance down the wall.

CHAPTER I

Introduction

Blood coagulation has been investigated quite extensively by many people with volumes of books written on the subject. Although the process of coagulation is a sequence of chemical reactions, chemistry fails to give a complete answer. The complete answer appears to involve hydrodynamics as evidenced by the recent investigations of Kantrowitz.[1] His experiments have shown that blood clotting occurs at the hydrodynamic stagnation point.

The stagnation point on a body in a flow is that point at which the velocities and the shear stress are zero. Immediately downstream of the stagnation point the flows are diverging from the point (Figure 1). Along the streamline which intersects the body at the stagnation point, the velocity component parallel to the body is zero.

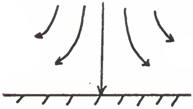


Figure 1. Flow

The equations governing the flow of an incompressible viscous fluid are equations derived by C. L. M. H. Navier and Sir G. G. Stokes.[2] They are derived for a Newtonian fluid in which the relationship between the stress tensor and the rate of strain tensor is linearized. Although blood is not a Newtonian fluid, near the wall the blood particles are generally absent so the fluid may be considered a plasma.

A solution of the Navier-Stokes equations was obtained by Hiemenz but the boundary conditions imposed are far from ideal.[3] The physical case would consist of a fluid flowing toward a body, such as a flat plate, which is then deflected by the plate. The solution of the Navier-Stokes equations is subjected to the boundary condition that at infinity the velocity component parallel to the wall is not zero. In other words, the flow starts out diverging from the stagnation streamline through no effect of the plate.

Additional methods used to investigate stagnation point flows involve the solution of an asymptotic form of the Navier-Stokes equations.[4, 5] The assumptions and the equations are developed in Chapter II, and it is there stated that at the stagnation point the assumptions of the theory are violated, thus rendering the solutions invalid.

Because of the evidence of Kantrowitz and others, it is felt that a thorough investigation of stagnation point flows is necessary. The problem is idealized by assuming a two-dimensional flow to be impinging at right angles onto a flat plate. The method to be used will be one put forward by G. F. Carrier and C. C. Lin.[6] It consists of obtaining a Stokes flow solution, which is valid near the stagnation point, and matching this solution to the boundary layer solution valid downstream of the stagnation point. The assumptions and the theory are presented in Chapter II.

In Chapter III the solution of the asymptotic equations valid downstream of the stagnation point is presented. The work was done by M. B. Glauert, and an exact solution was found.[7] The solution has been experimentally verified by P. Bakke with negligible error present between theory and experiment.[8]

Chapter IV is devoted to the solution for the flow in the neighborhood of the stagnation point. A first order correction is obtained and added to the solution. The solutions are obtained in terms of arbitrary constants which are evaluated by matching to the solution of M. B. Glauert. This is done in Chapter V, the results being contained in Chapter VI.

CHAPTER II

Theory and Method of Solution

A. Theory

In the analysis of an incompressible viscous flow, the use of a dimensionless parameter, the Reynolds number, permits the Navier-Stokes equations to be considered in asymptotic form, depending upon the magnitude of the Reynolds number. For large Reynolds numbers the Navier-Stokes equations become the asymptotic equations of L. Prandtl's boundary layer theory,[9] whereas for small Reynolds numbers the asymptotic equations describe a flow called Stokes flow because they were originally studied by Sir George G. Stokes.[10]

The Reynolds number is the ratio of the inertial forces to the viscous forces. It may be found from the relation Vd/ν , where V is the velocity of the fluid, d is a length characteristic of the body, and ν is the kinematic viscosity. For the case of stagnation point flow, the body is a flat plate of infinite extent, and thus has no fixed length which would characterize the body. As in flow over a flat plate, the length is taken as the distance measured along the plate from the origin of the coordinate system, chosen to be the stagnation point. Thus in stagnation point flow, the Reynolds number is a "local" one.

L. Prandtl hypothesized that at large Reynolds numbers the effects of viscous forces are confined to a narrow region close to the body. The solution of the flow problem consists of a solution of an asymptotic form of the Navier-Stokes equations. This solution is required to match the inviscid flow past the body at a short distance above the body. The equations are derived in many texts and are stated here for future reference.[11]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \qquad (2.1a)$$

$$\frac{\partial p}{\partial y} = 0 \quad , \qquad (2.1b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \qquad (2.1c)$$

The velocities have been made dimensionless with respect to a characteristic velocity, denoted by U, and the lengths by the ratio ν/U. The pressure is thus dimensionless with respect to ρU^2.

The solution of equations (2.1a), (2.1b), and (2.1c) for stagnation point flow by M. B. Glauert is presented in Chapter III.

Sir G. G. Stokes hypothesized that for very small Reynolds numbers the inertial forces in a fluid are negligible when compared to the viscous forces. In this problem the Reynolds number is small because both velocities and the physical length are small. The method of obtaining a solution consists of solving the Navier-Stokes equations in which the inertial terms have been deleted. Corrections to this solution are hopefully obtained by an iterative technique. Experimental investigations by D. Tritton show that the solution may be considered valid in any region in which the Reynolds number is less than one.[12]

The Navier-Stokes equations, from which the asymptotic equations are obtained, are derived in many texts and are, in cartesian coordinates[13]

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}} + \nu\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right), \qquad (2.2a)$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{y}} + \nu\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right), \qquad (2.2b)$$

and the continuity equation is

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 . \qquad (2.2c)$$

The left-hand side, which contains the squares of the velocities, is the inertial force on a fluid element and the right-hand side contains the pressure forces and the viscous forces. The $\bar{x}$ coordinate is the distance along the plate and the $\bar{y}$ coordinate is the distance perpendicular to the plate with the origin at the stagnation point. The $\bar{u}$ and $\bar{v}$ velocities are in the $\bar{x}$ and $\bar{y}$ directions respectively.

The variables can be made dimensionless, as in the boundary layer equations, by setting

$$\bar{u} = Uu,\ \bar{v} = Uv,\ \bar{x} = \nu x/U,\ \bar{y} = \nu y/U,\ \bar{\psi} = \nu\psi,\ \bar{p} = p\rho U^2$$

where ν is the kinematic viscosity, U is a reference velocity, and ψ is the stream function defined by

$$\bar{u} = \partial\bar{\psi}/\partial\bar{y},\quad \bar{v} = -\partial\bar{\psi}/\partial\bar{x}.$$

The elimination of the pressure from equations (2.2a) and (2.2b) allows the Navier-Stokes to be written

$$u\left(\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2 v}{\partial x^2}\right) + v\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x\partial y}\right) = \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 u}{\partial x^2\partial y} + \frac{\partial^3 v}{\partial x\partial y^2}. \qquad (2.3)$$

A coordinate transformation is made to make equation (2.3) more tractable. The transformation is to a complex plane given by

$$z = x + iy,\quad z^* = x - iy.$$

Upon introduction of the stream function and noting that

$$u + iv = -2i\,\partial\psi/\partial z^*, \qquad (2.4)$$

equation (2.3) becomes

$$\frac{\partial^4\psi}{\partial z^2\partial z^{*2}} = \frac{1}{2i}\left[\frac{\partial\psi}{\partial z^*}\frac{\partial^3\psi}{\partial z^2\partial z^*} - \frac{\partial\psi}{\partial z}\frac{\partial^3\psi}{\partial z\partial z^{*2}}\right]. \qquad (2.5)$$

The left-hand side now contains the viscous force terms and, using the notation of Carrier and Lin, will be denoted by L(ψ), whereas the right-hand side, which contains the inertial force terms, will be denoted by L*(ψ). It must be noted that although the asterisk denotes the complex conjugate of the coordinate z, it does not signify such in regards to the operator L.

B. Method of Solution

The analytical method to obtain the Stokes flow near the stagnation point is that put forward by Stokes.[14] It consists of an iterative scheme applied to equation (2.5). The first solution is that for which the inertial force terms have been deleted, i.e.,

$$L(\psi_0) = 0. \tag{2.6a}$$

Once this solution is obtained, the resulting corrections are obtained by solving the following equations in sequence:

$$L(\psi_1) = L^*(\psi_0) \tag{2.6b}$$

$$L(\psi_2) = L^*(\psi_0 + \psi_1) - L^*(\psi_0) \qquad (2.6b)$$

$$L(\psi_n) = L^*(\psi_0 + \ldots + \psi_{n-1}) - L^*(\psi_0 + \ldots + \psi_{n-2}) \tag{2.6n}$$

The above equations are developed in Chapter IV. Over any region in which the series

$$\psi = \psi_0 + \psi_1 + \ldots + \psi_n$$

converges, then this ψ will be taken as the solution of the Stokes flow.

Each of the equations (2.6a),...,(2.6n) can be integrated to within an arbitrary function. These functions will be evaluated by requiring that the solutions meet the imposed boundary conditions. The operator L is the biharmonic operator and, thus, the arbitrary functions must be biharmonic functions.

The solutions will be obtained in terms of arbitrary constants which will be evaluated by matching to the boundary layer solution. The method of matching will be discussed in Chapter V.

CHAPTER III

The Boundary Layer Solution

A. The Exact Solution

An exact solution of the boundary layer equations was given by M. B. Glauert.[15] His work is summarized in the following discussion.

A jet impinging on a flat plate is approximated by considering a jet in which the velocity distribution does not vary along its length. The jet is assumed to flow into a similar fluid at rest with respect to the plate. Sufficiently far from the jet the Reynolds number is large enough to validate a solution of Prandtl's asymptotic equations. If the pressure is assumed everywhere constant, the boundary layer equations (2.1a)-(2.1c) become

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}, \tag{3.1a}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \tag{3.1b}$$

The boundary conditions to be satisfied are

(1) $\psi = u = v = 0$ at $y = 0$

(2) $u \rightarrow 0$ as $y \rightarrow \infty$.

A similarity solution is sought for which the velocity u is assumed proportional to x^a and the jet thickness δ is assumed proportional to x^b. Substitution of the assumption into equation (3.1a) shows that a similarity solution exists provided that

$$a + 2b = 1. \tag{3.2}$$

A stream function is thus assumed,

$$\psi = x^{(1-b)} f(\eta), \tag{3.3a}$$

where

$$\eta = (1-b) y x^{-b}. \tag{3.3b}$$

With equations (3.3a) and (3.3b) the momentum equation (3.1a) becomes

$$f''' + ff'' + \alpha f'^2 = 0, \tag{3.4}$$

where

$$\alpha = \frac{2b-1}{1-b},$$

and the primes denote differentiation with respect to η.

The boundary conditions become

$$f(0) = f'(0) = 0 \text{ and } f'(\infty) = 0.$$

Integration of (3.4) between the limits of η and infinity yields

$$f'' + ff' - (\alpha-1) g = 0,$$

where

$$g = \int_\eta^\infty f'^2 d\eta. \tag{3.5}$$

Multiplication of this equation by f' and integration between the same limits yield

$$\frac{1}{2} f'^2 - fg + (\alpha-2) \int_\eta^\infty f' g \, d\eta = 0.$$

If no reverse flow is assumed to exist, then as η decreases, f will be zero at any point f' is zero. Since the above integral will then be non-zero, the equation requires that $\alpha = 2$ and thus

$$a = -1/2, \ b = 3/4.$$

An exact solution is found for equation (3.4) in terms of a parameter h. The solution is

$$f = h^2,$$

$$\eta = \log \frac{\sqrt{(1+h+h^2)}}{1-h} + \sqrt{3}\tan^{-1}\frac{\sqrt{3}h}{2+h}. \qquad (3.6)$$

With the flux of exterior momentum flux, F, which for the wall jet equals $\nu^2 U/40$, one can calculate the following flow quantities.

$$F = \nu^2 U/40 \qquad (3.7a)$$

$$\bar{\psi} = (40 F \nu \bar{x})^{1/4} f(\eta) \qquad (3.7b)$$

$$\bar{u} = \left(\frac{5F}{2\nu\bar{x}}\right)^{1/2} f'(\eta) \qquad (3.7c)$$

$$\eta = \left(\frac{5F}{32\nu^3\bar{x}^3}\right)^{1/4}\bar{y} \qquad (3.7d)$$

$$\tau_w/\rho = \frac{1}{9}\left(\frac{125F^3}{8\nu\bar{x}^5}\right)^{1/4} \qquad (3.7e)$$

B. The Series Solution

Anticipating that the Stokes flow solution will be a series, equation (3.6) will be unsuitable for matching on a term by term basis. A matching of this type will be possible only for a series solution of the boundary layer equations. Accordingly, assume f to be a power series of the form

$$f = \sum_{n=0}^{\infty} b_n \eta^n.$$

Equation (3.4) is readily written in terms of the power series.

$$\sum_{n=0}^{\infty} n(n-1)(n-2) b_n \eta^{n-3} + \sum_{n=0}^{\infty} b_n \eta^n \sum_{n=0}^{\infty} n(n-1) b_n \eta^{n-2} + 2\sum_{n=0}^{\infty} n b_n \eta^{n-1} \sum_{n=0}^{\infty} n b_n \eta^{n-1} = 0. \tag{3.8}$$

The expansion of the product series in equation (2.8) yields for the coefficient of η^n

$$(n+3)(n+2)(n+1) b_{n+3} + \sum_{l=0}^{n} (2n-l+1)(l+2) b_{l+2} b_{n-l} .$$

To have equation (2.8) identically equal to zero for any η it is necessary that the coefficients of η^n separately equal zero. This will be true provided that

$$b_{n+3} = - \frac{1}{(n+3)(n+2)(n+1)} \sum_{l=0}^{n} (2n-l+1)(l+2) b_{l+2} b_{n-l} ,$$

The boundary condition of zero stream function and zero velocity at the wall require $b_0 = b_1 = 0$. The coefficient b_2 will be the shear stress at the wall. Comparison of this solution with the exact solution, equation (2.6), shows that $b_2 = 1/9$. With these three coefficients all the rest may be found.

This solution, when transformed to the polar plane, shows that the terms appear as $r^{3/4}\theta^2$, $r^{5/4}\theta^5$, $r^{9/4}\theta^8$,... for small angles. The use of a small angle approximation was employed by Carrier and Lin. They thus assumed that although boundary layer theory implies large inertial forces, sufficiently close to the plate, the dominant forces are viscous.

CHAPTER IV

The Stokes Flow

A. Development of Solution Scheme and Boundary Conditions

As mentioned in Chapter II, the original assumption of Stokes was that for small Reynolds numbers the inertial forces are negligible. This is not to say they are zero; rather, that the velocities are small enough to neglect their squares. Assume the stream function to be composed of additive parts where the first term contains no inertial forces, the second contains them to the first order, etc., i.e.,

$$\psi = \psi_0 + \psi_1 + \dots + \psi_m . \tag{4.1}$$

Substitution of this series into equation (1.5) yields

$$\frac{\partial^4\left(\sum_{k=0}^{m}\psi_k\right)}{\partial z^2 \partial z^{*2}} = \frac{1}{2i}\left[\frac{\partial\left(\sum_{k=0}^{m}\psi_k\right)}{\partial z^*}\,\frac{\partial^3\left(\sum_{k=0}^{m}\psi_k\right)}{\partial z^2\partial z^*} - \frac{\partial\left(\sum_{k=0}^{m}\psi_k\right)}{\partial z}\,\frac{\partial^3\left(\sum_{k=0}^{m}\psi_k\right)}{\partial z\,\partial z^{*2}}\right],$$

or, in terms of the operators L and L*,

$$L\left(\sum_{k=0}^{m}\psi_k\right) = L^*\left(\sum_{k=0}^{m}\psi_k\right).$$

Since the function ψ_0 is assumed to contain no inertial effects, it will be a solution of the equation

$$L(\psi_0) = 0 . \tag{4.2}$$

This is the biharmonic equation making ψ_0 a biharmonic function. The remaining equation to be solved is

$$\frac{\partial^4 \psi_1}{\partial z^2 \partial z^{*2}} + \cdots + \frac{\partial^4 \psi_n}{\partial z^2 \partial z^{*2}} = L^*\Big(\sum_{i=0}^{n} \psi_i\Big).$$

The function ψ_1 may be eliminated from L(ψ) by setting $\partial^4 \psi_1 / \partial z^2 \partial z^{*2}$ equal to all the terms of L*(ψ) which are functions of ψ_0 only. It is obvious that the equation in this case is

$$L(\psi_1) = L^*(\psi_0). \tag{4.3}$$

ψ_2 may similarly be eliminated by setting $\partial^4 \psi_2 / \partial z^2 \partial z^{*2}$ equal to the remaining terms of L*(ψ) which are functions of $\psi_0 + \psi_1$ and ψ_1, i.e.,

$$2i \frac{\partial^4 \psi_2}{\partial z^2 \partial z^{*2}} = \frac{\partial(\psi_0 + \psi_1)}{\partial z^*} \frac{\partial^3 \psi_1}{\partial z^2 \partial z^*} + \frac{\partial \psi_1}{\partial z^*} \frac{\partial^3 \psi_0}{\partial z^2 \partial z^*} - \frac{\partial(\psi_0 + \psi_1)}{\partial z} \frac{\partial^3 \psi_1}{\partial z \partial z^*} - \frac{\partial \psi_1}{\partial z} \frac{\partial^3 \psi_0}{\partial z \partial z^{*2}}. \tag{4.4a}$$

Addition and subtraction of the terms

$$\frac{\partial \psi_0}{\partial z^*} \frac{\partial^3 \psi_0}{\partial z^2 \partial z^*} \quad \text{and} \quad \frac{\partial \psi_0}{\partial z} \frac{\partial^3 \psi_0}{\partial z \partial z^{*2}}$$

permits writing equation (4.4a) in terms of the operators;

$$L(\psi_2) = L^*(\psi_0 + \psi_1) - L^*(\psi_0). \tag{4.4b}$$

Continuing in like manner the n-th part of the stream function will be found from the equation

$$L(\psi_n) = L^*\Big(\sum_{i=0}^{n-1} \psi_i\Big) - L^*\Big(\sum_{i=0}^{n-2} \psi_i\Big).$$

This method will yield the solution

$$\psi = \psi_0 + \psi_1 + \dots + \psi_n$$

over any region in which the series is convergent and satisfies the original assumptions of the theory.

A solution for each of the parts of the stream function can be found to within an arbitrary biharmonic function. Denoting the particular integral by ψ_{i0} and the arbitrary function by ψ_{ij}, the solution of any of the equations may be written

$$\psi_i = \psi_{i0} + \psi_{ij} .$$

The coordinate systems to be used to obtain the solutions of the equations is the z, z^* plane, while the boundary conditions are more readily applied in the polar plane, with the solution plotted in the cartesian plane (Figure 2).

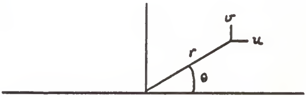


Figure 2. Coordinate System

With the stagnation point located at (0,0), the boundary conditions are

(1) $\psi_i = 0$ at $\theta = 0, \pi$

(2) $u + iv = -2i\, \partial \psi_i / \partial z^* = 0$ at $\theta = 0, \pi$

(3) $u = \mathrm{Re}(-2i\, \partial \psi_i / \partial z^*) = 0$ at $\theta = \pi/2$.

B. Solution With No Inertial Forces

The first approximation to the flow is given by the solution of equation (4.2). E. Goursat[16] gave a solution of the equation in terms of two arbitrary functions $\varphi(z)$ and $\chi(z)$ and their conjugates $\varphi^*(z)$ and $\chi^*(z)$. It was found to be

$$\psi_0 = Re\left\{z^*\varphi(z) + \chi(z)\right\}$$

which can be written in a more convenient form, viz.,

$$\psi_0 = z^*\varphi(z) + z\,\varphi^*(z) + \chi(z) + \chi^*(z). \qquad (4.5)$$

The series solution of the boundary layer equation resulted in the terms being multiples of the fourth root of the independent variables r and θ. Accordingly, the function $\varphi(z)$ is assumed to vary in like manner. A general expression can be written for $\varphi(z)$ in terms of the infinite series

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^{\frac{n+1}{4}}.$$

Upon substitution of this series into equation (4.5), it is found that the first boundary condition will be satisfied for

$$\chi(z) = -\sum_{n=0}^{\infty} a_n z^{\frac{n+5}{4}}.$$

The general solution may thus be written

$$\psi_0 = z^*\sum_{n=0}^{\infty} a_n z^{\frac{n+1}{4}} + z\sum_{n=0}^{\infty} a_n z^{*\frac{n+1}{4}} - \sum_{n=0}^{\infty} a_n z^{\frac{n+5}{4}} - \sum_{n=0}^{\infty} a_n z^{*\frac{n+5}{4}}. \qquad (4.6)$$

The velocities are found using equation (2.4) which states that u + iv = -2i $\partial \psi_0 / \partial z^*$. Upon differentiation and transformation to the polar plane, they may be found by the relation

$$u+iv = -2i\sum_{m=0}^{\infty} a_n r^{\frac{n+1}{4}}\left\{\left(\frac{n+1}{4}\right)e^{-\left(\frac{n+1}{4}\right)i\theta}\left[e^{2i\theta}-1\right]+2i\sin\left(\frac{n+1}{4}\right)\theta\right\}.$$

The boundary condition requiring zero velocities at $\theta=0$ is identically satisfied, but at $\theta=\pi$

$$u+iv \propto \sin\left(\frac{n+1}{4}\right)\pi.$$

At $\theta=\pi$ the boundary condition can be satisfied if (n+1)/4 = m, m = 0,1,2,.... The third boundary condition produces a similar situation in that at $\theta=\pi/2$

$$u \propto \sin(m-2)\pi/2 .$$

As before, this boundary condition will be satisfied provided that (m-2)/2 = l, l = 0,1,2,....

A solution to the Navier-Stokes equations, in which the inertial forces have been neglected and which satisfies the boundary conditions of stagnation point flow, is

$$\psi_0 = \sum_{l=0}^{\infty} 2a_l r^{2l+3}\left[\cos(2l+1)\theta - \cos(2l+3)\theta\right], \tag{4.7a}$$

with

$$u = 4\sum_{l=0}^{\infty} a_l r^{2l+2}\Big\{\sin(2l+2)\theta + (l+1)\big[\sin 2\theta \cdot(\cos 2l\theta - \sin 2l\theta) - \sin 2l\theta\big]\Big\}, \tag{4.7b}$$

and

$$v = -8\sum_{l=0}^{\infty}(l+1)a_l r^{2l+2}\sin\theta\sin(2l+1)\theta . \tag{4.7c}$$

C. The First Iteration

A solution is sought for equation (4.3) subject to the mathematical and boundary conditions stated in Part A of this chapter. The particular integral, ψ_{10}, will be obtained and the biharmonic functions, ψ_{1j}, will be evaluated so as to make the boundary values correct.

Substitution of equation (4.3a) into equation (4.3) results in

$$\frac{\partial^4\psi_{10}}{\partial z^2\partial z^{*2}} = \frac{1}{2i}\Big\{\Big[\sum_{l=0}^{\infty} a_l z^{2l+2} + z\sum_{l=0}^{\infty}(2l+2)a_l z^{*2l+1}$$

$$-\sum_{l=0}^{\infty}(2l+3)a_l z^{*2l+2}\Big]\sum_{l=0}^{\infty}(2l+2)(2l+1)a_l z^{2l} - \Big[z^*\sum_{l=0}^{\infty}(2l+2)a_l z^{2l+1}$$

$$+\sum_{l=0}^{\infty} a_l z^{*2l+2} - \sum_{l=0}^{\infty}(2l+3)a_l z^{2l+2}\Big]\sum_{l=0}^{\infty}(2l+2)(2l+1)a_l z^{*2l}\Big\}.$$

The integration is readily performed and the particular integral, ψ_{10}, when transformed to the polar plane, becomes

$$\psi_{10} = \frac{1}{2}\sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty}\frac{(2l+2)(2l+1)}{(2l+2k+3)(2l+2k+4)} a_l r^{2l+2k+6} \sin 2(l+k+1)\theta$$

$$+\sum_{k=0}^{\infty}\frac{a_k}{(2k+3)}\sum_{l=0}^{\infty}\left(\frac{2l+1}{2l+3}\right)a_l r^{2l+2k+6} \sin 2(l-k)\theta \qquad (4.8)$$

$$-\sum_{k=0}^{\infty}\frac{a_k}{(2k+4)}\sum_{l=0}^{\infty} a_l r^{2l+2k+6} \sin 2(l-k-1)\theta.$$

This particular integral satisfies boundary conditions (1) and (3); that is, zero mass flow through the wall and u = 0 at $\theta = \pi/2$, but at the wall

$$(u+iv)_{\theta=0} = -\sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l r^{2l+2k+5} \left[\frac{l-k-1}{k+2} + 2\frac{(k-l)(2l+1)}{(2l+3)(2k+3)} - \frac{1}{2}\left(\frac{l+k+1}{l+k+2}\right)\frac{(2l+2)(2l+1)}{(2l+2k+3)} \right], \tag{4.9}$$

with

$$(u+iv)_{\theta=\pi} = -(u+iv)_{\theta=0} ,$$

Thus, the arbitrary function must be evaluated to make the velocities zero at the wall.

The general form for a biharmonic function was given in equation (4.5). Three possibilities exist for satisfying the boundary condition (2):

(1) $\chi(z) = 0, \quad \varphi(z) \neq 0,$

(2) $\varphi(z) = 0, \quad \chi(z) \neq 0,$

(3) $\varphi(z), \quad \chi(z) \neq 0.$

Case (1): $\chi(z) = 0$

Assume the biharmonic function, ψ_{11}, to be given by

$$2i\psi_{11} = z^* \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l A_{kl} z^{2l+2k+5} - z \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l A_{kl} z^{*\,2l+2k+5} .$$

It is readily found that

$$(u+iv)_{\theta=0} = -\sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l A_{kl} r^{2l+2k+5} \left[-2(l+k+2)\right],$$

with

$$(u+iv)_{\theta=\pi} = -(u+iv)_{\theta=0} .$$

Thus ψ_{11}, when added to ψ_{10}, will result in zero velocities at the wall provided

$$A_{kl} = \frac{1}{2(l+k+2)}\left[\frac{l-k-1}{k+2} + 2\frac{(k-l)(2l+1)}{(2l+3)(2k+3)} - \frac{1}{2}\left(\frac{l+k+1}{l+k+2}\right)\frac{(2l+2)(2l+1)}{(2l+2k+3)}\right].$$

Case (2): $\varphi(z) = 0$

The biharmonic function, denoted by ψ_{12}, may be written

$$2i\psi_{12} = \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l B_{kl}\, z^{2l+2k+6} - \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l B_{kl}\, z^{*\,2l+2k+6}.$$

The velocities at the wall are found to be

$$(u+iv)_{\theta=0} = -\sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l B_{kl}\, r^{2l+2k+5}\left[-2(l+k+3)\right]$$

with

$$(u + iv)_{\theta=\pi} = -(u + iv)_{\theta=0}.$$

This function will adjust the value of ψ_{10} to satisfy boundary condition (2) at the wall if

$$B_{kl} = \frac{1}{2(l+k+3)}\left[\frac{l-k-1}{k+2} + 2\frac{(k-l)(2l+1)}{(2l+3)(2k+3)} - \frac{1}{2}\left(\frac{l+k+1}{l+k+2}\right)\frac{(2l+2)(2l+1)}{(2l+2k+3)}\right].$$

Case (3): $\varphi(z),\ \chi(z) \neq 0$

This biharmonic function, ψ_{13}, is essentially a linear combination of cases (1) and (2), with the difference being in the constant.

$$2i\psi_{13} = \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l C_{kl}\, z^{2l+2k+6} - \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l C_{kl}\, z^{*\,2l+2k+6}$$

$$+ z^{*}\sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l C_{kl}\, z^{2l+2k+5} - z\sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l C_{kl}\, z^{*\,2l+2k+5}.$$

The velocities at the wall are given by

$$(u+iv)_{\theta=0} = -\sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l C_{kl}\, r^{2l+2k+5}\left[-2(l+k+5)\right],$$

with

$$(u + iv)_{\theta=\pi} \quad = \quad -(u + iv)_{\theta=0} \; .$$

As before, the solution will sum to zero provided that

$$C_{kl} = \frac{1}{2(l+k+5)}\left[\frac{l-k-1}{k+2} + 2\frac{(k-l)(2l+1)}{(2l+3)(2k+3)} - \frac{1}{2}\left(\frac{l+k+1}{l+k+2}\right)\frac{(2l+2)(2l+1)}{(2l+2k+3)}\right].$$

Each of the biharmonic functions, ψ_{1j}, j = 1,2,3, satisfy the boundary conditions (1) and (3) and thus produce no change in their values.

A solution for stagnation point flow, for which a first order correction of the inertial effects has been made, thus becomes

$$\psi = \sum_{l=0}^{\infty} 2a_l r^{2l+3}\left[\cos(2l+1)\theta - \cos(2l+3)\theta\right]$$

$$+\frac{1}{2}\sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} \frac{(2l+2)(2l+1)}{(2l+2k+3)(2l+2k+4)} a_l\, r^{2l+2k+6} \sin 2(l+k+1)\theta$$

$$+ \sum_{k=0}^{\infty} \frac{a_k}{(2k+3)} \sum_{l=0}^{\infty} \left(\frac{2l+1}{2l+3}\right) a_l\, r^{2l+2k+6} \sin 2(l-k)\theta$$

$$- \sum_{k=0}^{\infty} \frac{a_k}{(2k+4)} \sum_{l=0}^{\infty} a_l\, r^{2l+2k+6} \sin 2(l-k-1)\theta$$

$$+ \psi_{1j} ,$$

where

$$\psi_{11} = \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l A_{kl} r^{2l+2k+6} \sin 2(l+k+2)\theta,$$

$$\psi_{12} = \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l B_{kl} r^{2l+2k+6} \sin 2(l+k+3)\theta,$$

or

$$\psi_{13} = \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} a_l C_{kl} r^{2l+2k+6} \left[\sin 2(l+k+2)\theta + \sin 2(l+k+3)\theta\right],$$

with A_{kl}, B_{kl}, and C_{kl} as previously defined.

The solution of equation (4.4b) results in terms beginning with the ninth power of r with the coefficient being

$$\text{const. (times) } a_i a_j a_k \ (10^{-3}).$$

Since the series is presumed convergent, it is felt that these terms will be negligible compared to those already calculated and, thus, the iterative scheme is terminated at this point.

CHAPTER V

The Matching Procedure

The arbitrary constants obtained in Chapter IV must be evaluated subject to some boundary condition. The idea to be employed here is that used by Carrier and Lin and Imai.[6,17] Rather than require the solution to have some velocity distribution at infinity, the flow will be required to "carry over" into the boundary layer solution.

This method of direct comparison of two asymptotic solutions is subject to a large number of questions for which there are very few answers. In the final analysis, experimentation must provide the answer as to where this "carry over" can occur, if such a region does indeed exist. Some of the terms of the Carrier and Lin solution matched Blasius' boundary layer solution when the small angle approximation was made. In this problem, however, the term containing the sixth power of r has the factor θ^{23} compared to θ^{8} .

Van Dyke[18] states several requisites for matching different parameter expansions. Examples are, that one of the solutions be convergent over the whole region, or that some similarity rule be followed. This solution does not satisfy either of them, as the Stokes flow solution is not complete, and a similarity rule is not applicable.

It may be inferred from the discussion by Van Dyke that unless the above are satisfied, the Stokes flow solution should be matched to some intermediate solution. An intermediate solution, between Stokes flow and boundary layer flow, would be a solution either of the equations of Oseen[19] or the full Navier-Stokes equations. Oseen flow is essentially a perturbation of Stokes flow; and, for this problem, was found to be of no use. It should be remembered that there is a solution of the Navier-Stokes equations for this type of problem, but the boundary conditions at infinity are not consistent with the particular physical situation.

The method to be used here is similar to that used by Imai.[20] He matched the skin friction at a Reynolds number of one for the flow over a flat plate. For the present problem a fixed location cannot be defined at which the Reynolds number is one.

In the experimental investigation of M. B. Glauert's solution, it was found that at 14.3 cm. from the stagnation point, the difference between theory and experiment was negligible.[21] There is no mention of measurements closer than 14.3 cm., so the error introduced by matching at smaller distances from the origin cannot be determined. Other points will be picked at which to match in order that the effect on location can be ascertained.

For each constant which is retained it is necessary to differentiate the stream function. Since the solution of equation (4.4b) would begin with terms beginning at r^9, the series retained must terminate with r^8, the r^9 term being incomplete. This terminated series will contain three arbitrary constants, a_i, $i = 0,1,2$. Since the zeroth and first derivatives are identically zero at the plate, the second through the fourth derivatives are necessary.

The partial operators, which when evaluated at the wall, become

$$\left(\frac{\partial}{\partial y}\right)_{y=0} = \left(\frac{1}{r}\frac{\partial}{\partial \theta}\right)_{\theta=0},$$

$$\left(\frac{\partial^2}{\partial y^2}\right)_{y=0} = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)_{\theta=0},$$

$$\left(\frac{\partial^3}{\partial y^3}\right)_{y=0} = \left(\frac{1}{r^3}\frac{\partial^3}{\partial \theta^3} + \frac{3}{r^2}\frac{\partial^2}{\partial r \partial \theta} - \frac{2}{r^3}\frac{\partial}{\partial \theta}\right)_{\theta=0},$$

and

$$\left(\frac{\partial^4}{\partial y^4}\right)_{y=0} = \left(\frac{1}{r^4}\frac{\partial^4}{\partial \theta^4} + \frac{6}{r^3}\frac{\partial^3}{\partial r \partial \theta} - \frac{8}{r^4}\frac{\partial^2}{\partial \theta^2} + \frac{3}{r^2}\frac{\partial^2}{\partial r^2} - \frac{3}{r^3}\frac{\partial}{\partial r}\right)_{\theta=0}.$$

The truncated series is

$$\psi = 2a_0 r^3(\cos\theta - \cos 3\theta) + 2a_1 r^5(\cos 3\theta - \cos 5\theta) + 2a_2 r^7(\cos 5\theta - \cos 7\theta) + \frac{1}{3}a_0^2 r^6 \sin 2\theta + \left(\frac{2}{5}\sin 4\theta + \frac{2}{15}\sin 2\theta\right)a_0 a_1 r^8 + \psi_{1j}.$$

where

$$\psi_{11} = a_0^2 A_{00} r^6 \sin 4\theta + (A_{10} + A_{01})a_0 a_1 r^8 \sin 6\theta,$$

$$\psi_{12} = a_0^2 B_{00} r^6 \sin 6\theta + (B_{10} + B_{01}) a_0 a_1 r^8 \sin 8\theta,$$

and

$$\psi_{13} = (\sin 4\theta + \sin 6\theta) a_0^2 C_{00} r^6 + (C_{10} + C_{01})(\sin 6\theta + \sin 8\theta) a_0 a_1 r^8.$$

Differentiation of the series solution for the boundary layer flow shows that

$$\psi_{yy}\Big)_{\theta=0} = b_2/8r^{5/4},$$

$$\psi_{yyy}\Big)_{\theta=0} = 0,$$

and

$$\psi_{yyyy}\Big)_{\theta=0} = 0,$$

where b_2 is found to be 1/9.

The values of the constants are contained in the next chapter.

CHAPTER VI

Results and Conclusions

The locations of the matching points are determined by considering the properties of blood and the experimental evidence of Bakke. Blood has been found to have a kinematic viscosity of 3.8(10^{-2}) cm.^{2}/sec. with its average velocity in blood vessels to be in the range of 30.0 to 40.0 cm./sec.[22] Choosing for convenience the reference velocity, U, to be 38.0 cm./sec., the non-dimensionalizing factor for the lengths becomes 1(10^{-3}) cm. With the experimental verification of the boundary layer solution at 14.3 cm., the first matching point chosen is 14,300. Other points chosen correspond to 1.0, 5, 0, and 10.0 cm. The solutions are plotted in Figures 3-6. Figure 7 shows the solution for matching at 14.3 cm. on a diminished scale. The values for the coefficients are contained in Table 1.

The solutions are plotted only for the first case, i.e., $\chi(x) = 0$. Examination of the data shows that in the region about the stagnation point, ψ_{11} is consistently about 93.7 percent of ψ_{12} and 94.7 percent of ψ_{13} for the matching at 14.3 cm. For the other matching points ψ_{11} is never less than 92.6 percent of ψ_{12}, or 91.4

TABLE 1
VALUES FOR THE CONSTANTS AT THE MATCHING LOCATIONS

CASE	CONSTANTS	LOCATION OF MATCHING POINTS - $X(10^{-3})$			
		1.0	5.0	10.0	14.3
1	a_0	$2.133(10^{-10})$	$5.706(10^{-12})$	$1.199(10^{-12})$	$5.364(10^{-13})$
	a_1	$-4.210(10^{-17})$	$-4.505(10^{-20})$	$-2.367(10^{-20})$	$-5.177(10^{-22})$
	a_2	$8.420(10^{-24})$	$3.604(10^{-28})$	$4.735(10^{-30})$	$5.064(10^{-31})$
2	a_0	$2.276(10^{-10})$	$6.089(10^{-12})$	$1.280(10^{-12})$	$5.724(10^{-13})$
	a_1	$-5.233(10^{-17})$	$-5.599(10^{-20})$	$-2.943(10^{-20})$	$-6.435(10^{-22})$
	a_2	$1.046(10^{-23})$	$4.479(10^{-28})$	$5.885(10^{-30})$	$6.294(10^{-31})$
3	a_0	$2.264(10^{-10})$	$6.056(10^{-12})$	$1.273(10^{-12})$	$5.694(10^{-22})$
	a_1	$-5.146(10^{-12})$	$-5.506(10^{-20})$	$-2.943(10^{-20})$	$-6.328(10^{-22})$
	a_2	$1.029(10^{-23})$	$4.405(10^{-28})$	$5.787(10^{-30})$	$6.189(10^{-31})$

percent of ψ_{13} in the region of interest. Differences in the graphs would not be detectable except on a greatly expanded scale.

Lines of constant Reynolds numbers are also plotted to determine the region of validity for the Stokes flow solution. The Reynolds number can be viewed as the ratio of the inertial forces to the viscous forces; thus, because of the assumption that inertial forces are negligible in Stokes flow, the region of validity is that region in which the Reynolds number is less than one. It appears at first that for very small distances from the origin, the Reynolds number would be small regardless of the velocities. The assumption of negligible squares of the velocities would be violated even though the Reynolds number is small. A choice of the vertical distance rather than the horizontal distance as the characteristic length might be an alternative, but, for the solutions obtained, the regions graphed are all at a Reynolds number less than one. The choice of the horizontal distance is more restrictive in defining the region.

If the limit of the validity of Stokes flow is chosen at a Reynolds number of .01, the region for matching at 1.0 cm. is essentially a semicircle of radius 012 cm., while for the matching at 14.3 cm., the region is essentially a semicircle of radius 1.4-1.6 cm. The reason is quite apparent when the boundary layer solution is considered. The solutions were matched by requiring the shear stress and its derivatives to be equal at some point. The shear stress in the boundary layer flow is proportional to $r^{-5/4}$. Analytically, the shear stress is directly proportional to the velocity gradient, so that as the shear stress decreases, near the wall there is less mass flow.

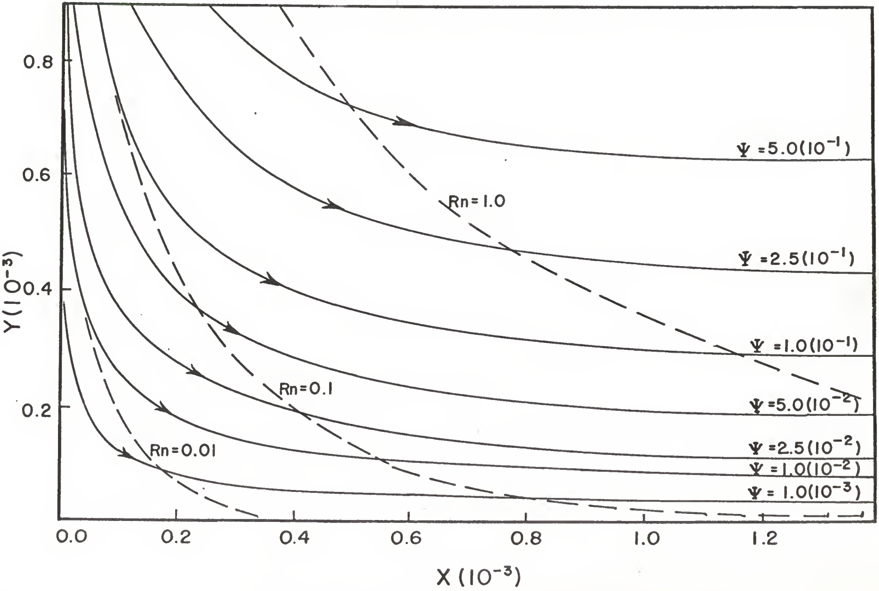
Beyond a Reynolds number of about unity, the flow begins to depart from that which had been expected; but this region is certainly well beyond that at which a Stokes flow solution could be considered valid.

The assumption of Chapter IV, that the second iteration is negligible, is borne out by examining Table 1. In Chapter IV it was stated that the first term of this second iteration contained a triple product of the constants. Even if the product is a_0^3, the term will be of order 10^{-33}, which can be considered negligible compared to a_0a_1.

Figure 8 shows the variation of the shear stress at the wall with the distance down the wall for the matching at 14.3 cm. The shear stress has been made dimensionless with respect to ρU^2. The variation is shown for the two cases of $\phi(z) = 0$ and $\chi(z) = 0$. The third case falls between these values. The variation is essentially linear with both cases having values of $8.88(10^{-6})$ at the match point.

The stagnation point flow can thus be considered as Stokes flow up to the matching location and then boundary layer flow. Figure 6 gives the flow which will produce the experimentally verified boundary layer flow.

Figure 3. Streamlines for the Stokes Flow Calculated by Requiring that the Skin Friction and Two Subsequent Derivatives be Equal for the Stokes Flow and the Boundary Layer Flow at 1.0 Centimeters. The Dashed Lines are Lines of Constant Reynolds Number Based on the Distance Down the Wall and the Local Velocity.



$\Psi = 5.0(10^{-1})$
$\Psi = 2.5(10^{-1})$
$\Psi = 1.0(10^{-1})$
$\Psi = 5.0(10^{-2})$
$\Psi = 2.5(10^{-2})$
$\Psi = 1.0(10^{-2})$
$\Psi = 1.0(10^{-3})$
Rn=1.0
Rn=0.1
Rn=0.01
0.8
0.6
0.4
0.2
$Y(10^{-3})$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
$X(10^{-3})$

Figure 4. Streamlines for the Stokes Flow Calculated by Requiring that the Skin Friction and Two Subsequent Derivatives be Equal for the Stokes Flow and the Boundary Layer Flow at 5.0 Centimeters. The Dashed Lines are Lines of Constant Reynolds Number Based on the Distance Down the Wall and the Local Velocity.

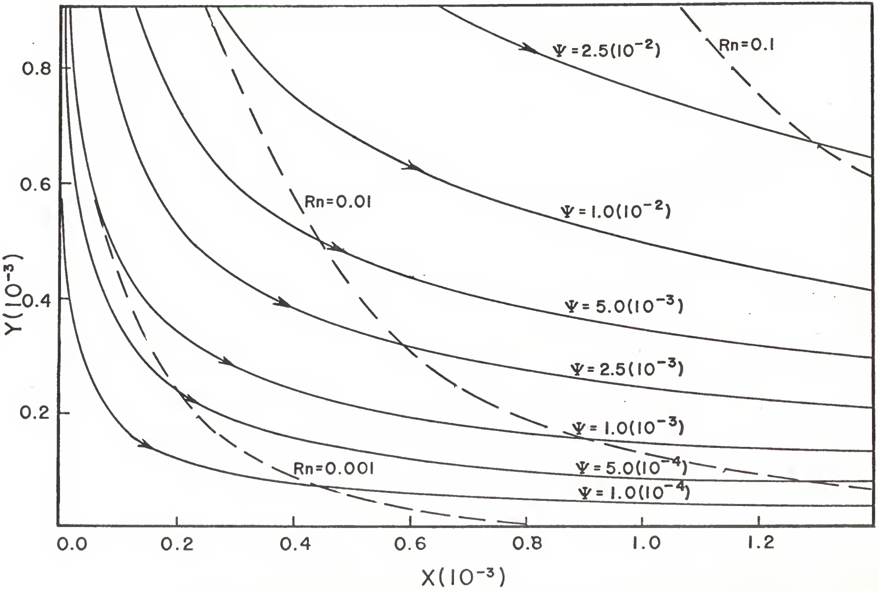

Ψ=2.5(10^-2)
Rn=0.1
Rn=0.01
Ψ=1.0(10^-2)
Ψ=5.0(10^-3)
Ψ=2.5(10^-3)
Ψ=1.0(10^-3)
Rn=0.001
Ψ=5.0(10^-4)
Ψ=1.0(10^-4)
0.8
0.6
0.4
0.2
Y(10^-3)
0.0
0.2
0.4
0.6
0.8
1.0
1.2
X(10^-3)

Figure 5. Streamlines for the Stokes Flow Calculated by Requiring that the Skin Friction and Two Subsequent Derivatives be Equal for the Stokes Flow and the Boundary Layer Flow at 10.0 Centimeters. The Dashed Lines are Lines of Constant Reynolds Number Based on the Distance Down the Wall and the Local Velocity.

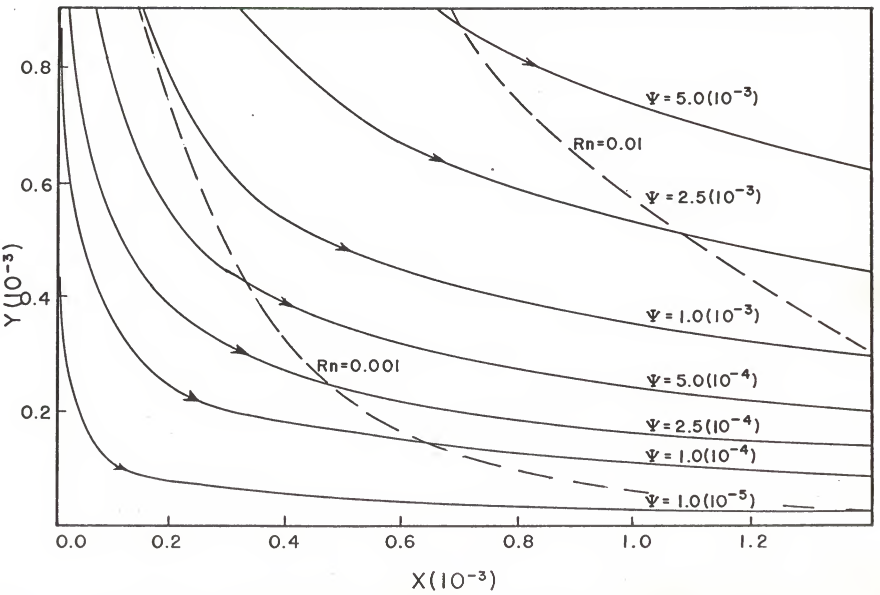
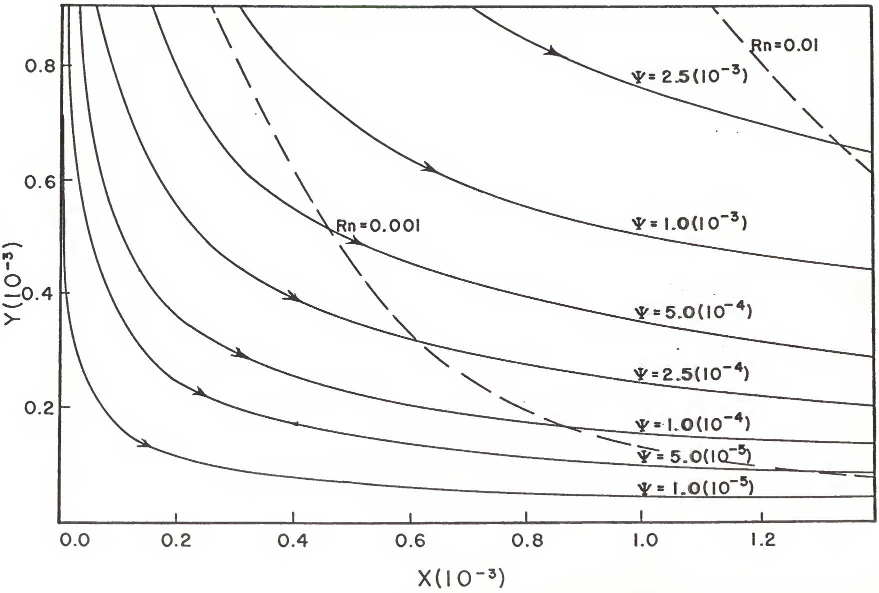

$\Psi = 5.0(10^{-3})$
Rn=0.01
$\Psi = 2.5(10^{-3})$
$\Psi = 1.0(10^{-3})$
Rn=0.001
$\Psi = 5.0(10^{-4})$
$\Psi = 2.5(10^{-4})$
$\Psi = 1.0(10^{-4})$
$\Psi = 1.0(10^{-5})$
0.8
0.6
0.4
0.2
$Y(10^{-3})$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
$X(10^{-3})$

Figure 6. Streamlines for the Stokes Flow Calculated by Requiring that the Skin Friction and Two Subsequent Derivatives be Equal for the Stokes Flow and the Boundary Layer Flow at 14.3 Centimeters. The Dashed Lines are Lines of Constant Reynolds Number Based on the Distance Down the Wall and the Local Velocity.


Rn=0.01
$\Psi = 2.5(10^{-3})$
$\Psi = 1.0(10^{-3})$
Rn=0.001
$\Psi = 5.0(10^{-4})$
$\Psi = 2.5(10^{-4})$
$\Psi = 1.0(10^{-4})$
$\Psi = 5.0(10^{-5})$
$\Psi = 1.0(10^{-5})$
$Y(10^{-3})$
0.8
0.6
0.4
0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
$X(10^{-3})$

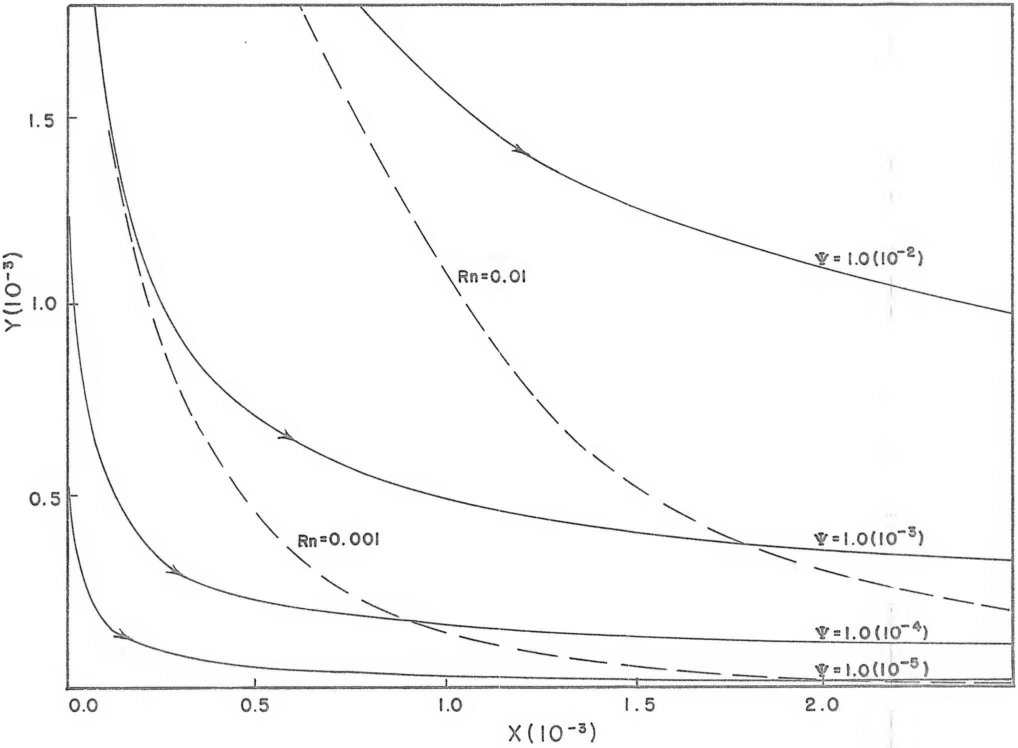


FIGURE 7

STREAMLINES FOR MATCHING AT 14.3 CM

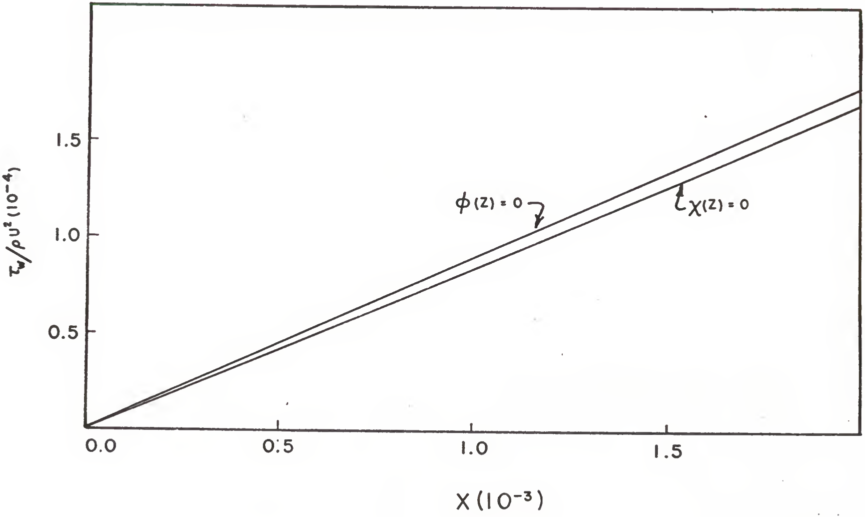


FIGURE 8

VARIATION OF SHEAR STRESS AT THE WALL WITH DISTANCE

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BIOGRAPHICAL SKETCH

Bruce Boyd Henriksen was born December 4, 1941, at Milwaukee, Wisconsin. In 1947, he moved to Ormond Beach, Florida. He attended public schools in Ormond Beach and Daytona Beach, Florida. In 1959, he was graduated from Seabreeze High School in Daytona Beach, Florida. From 1959, to 1964, he attended the University of Florida, and received the degree of Bachelor of Science in Education. In the fall of 1964, he entered the Graduate School of the University of Florida and worked as a graduate research assistant until December, 1965. In January, 1966, he was awarded a NASA Traineeship, and in April, 1966, he was awarded the Master of Science in Engineering with a major in Aerospace Engineering. He was awarded the degree of Doctor of Philosophy with a major in Aerospace Engineering in August, 1968.

He is married to the former Mary Barbara Kohrt of Daytona Beach, Florida, and is the father of one son and one daughter.

This disseration was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1968

Dean, College of Engineering

Dean, Graduate School

Supervisory Committee:

Knox Millsaps
Chairman

B M Leadon

M. H. Clarkson

O. E. Myers

C. W. Nelson